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ON THE EXTENSION OF THE EXPONENTIAL THEOREM.[*]

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§1. INTRODUCTION.

In our algebra,[†] we define

$$\underset{n \doteq \infty}{\mathrm{L}} \left(1+\frac{x}{n}\right)^{n} = F(x),$$

for all values of x and n real or complex, and denote by $f(x)$ the value which $F(x)$ takes when n is a positive integer and x is real. We prove that

$$f(x)=1+x+\frac{x^2}{2\,!}+\frac{x^3}{3\,!}+\ldots=e^x.$$

Next we prove that $F(x)=f(x)$ for any real value of x, in whatever way n as a real number becomes indefinitely great. Then we prove for a positive integral n and a complex x, which is called z, that $F(z)=f(z)$. Then we state, p. 240: "The extension to any real value of n may be made by considering the limits of the moduli and arguments of

$$(1+\frac{z}{m+1})^m,\quad (1+\frac{z}{n})^n,\quad (1+\frac{z}{m})^{m+1},$$

and finally extending to a negative n."

It is the object of the present paper to furnish the details of this statement.

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[*] Presented to the American Mathematical Society, April 24, 1909.

[†] *College Algebra*, Metzler, Roe and Bullard (Longmans, Green and Co.), 1908.

§2. EXTENSION TO ANY POSITIVE n.

For convenience we divide the proof into three cases:

1. Where $90^\circ \geqq \arg z > 0$,
2. Where $180^\circ > \arg z > 90^\circ$,
3. Where $360^\circ > \arg z > 180^\circ$.

§3. CASE 1. $90^\circ \geqq$ ARG $z > 0$.

Here $z = \frac{MP}{OM} = \frac{MP}{1}$. And m being any integer we may always have

$$m+1 > n > m$$

and since $z = r(\cos\phi + i\sin\phi)$ we have, Fig. 1,

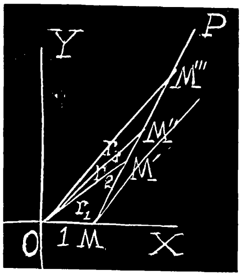

Fig. 1.

$$\frac{z}{m+1} = \frac{MM'}{OM} = \frac{MM'}{1} = \frac{r}{m+1}(\cos\phi + i\sin\phi),$$

$$\frac{z}{n} = \frac{MM''}{OM} = \frac{MM''}{1} = \frac{r}{n}(\cos\phi + i\sin\phi),$$

$$\frac{z}{m} = \frac{MM'''}{OM} = \frac{MM'''}{1} = \frac{r}{m}(\cos\phi + i\sin\phi),$$

where length $MM' = \frac{r}{m+1}$, length $MM'' = \frac{r}{n}$, length $MM''' = \frac{r}{m}$. We also have

$$1 + \frac{z}{m+1} = \frac{OM'}{1},\ \operatorname{mod}\left(1 + \frac{z}{m+1}\right) = \text{length } OM' = r_1,$$

$$MOM' = \arg\left(1 + \frac{z}{m+1}\right) = \phi_1,$$

$$1 + \frac{z}{n} = \frac{OM''}{1},\ \operatorname{mod}\left(1 + \frac{z}{n}\right) = \text{length } OM'' = r_2,\ MOM'' = \arg\left(1 + \frac{z}{n}\right) = \phi_2,$$

$$1 + \frac{z}{m} = \frac{OM'''}{1},\ \operatorname{mod}\left(1 + \frac{z}{m}\right) = \text{length } OM''' = r_3,\ MOM''' = \arg\left(1 + \frac{z}{m}\right) = \phi_3.$$

And if $90^\circ \geqq \arg z > 0$, $r_1 < r_2 < r_3$, $\phi_1 < \phi_2 < \phi_3$, whence

$$r^m{}_1 < r^n{}_2 < r^{m+1}{}_3,\quad m\phi_1 < n\phi_2 < (m+1)\phi_3;$$

that is

$$\text{mod}\,(1+\frac{z}{m+1})^m<\text{mod}\,(1+\frac{z}{n})^n<\text{mod}\,(1+\frac{z}{m})^{m+1}, \tag{1}$$

$$\text{arg}\,(1+\frac{z}{m+1})^m<\text{arg}\,(1+\frac{z}{n})^n<\text{arg}\,(1+\frac{z}{m})^{m+1}. \tag{2}$$

Now $\underset{m\doteq\infty}{\text{L}}(1+\frac{z}{m+1})^m=\underset{m\doteq\infty}{\text{L}}(1+\frac{z}{m})^{m+1}$, since

$$\underset{m\doteq\infty}{\text{L}}(1+\frac{z}{m+1})^m=\frac{\underset{m\doteq\infty}{\text{L}}(1+\frac{z}{m+1})^{m+1}}{\underset{m\doteq\infty}{\text{L}}(1+\frac{z}{m+1})}=f(z),\text{ and}$$

$$\underset{m\doteq\infty}{\text{L}}(1+\frac{z}{m})^{m+1}=\underset{m\doteq\infty}{\text{L}}(1+\frac{z}{m})^m\,(1+\frac{z}{m})=f(z).$$

Hence, since when two complex numbers are equal their moduli and arguments are equal,

$$\underset{m\doteq\infty}{\text{L}}\text{mod}\,(1+\frac{z}{m+1})^m=\underset{m\doteq\infty}{\text{L}}\text{mod}\,(1+\frac{z}{m})^{m+1}=\text{mod}\,f(z),$$

$$\underset{m\doteq\infty}{\text{L}}\text{arg}\,(1+\frac{z}{m+1})^m=\underset{m\doteq\infty}{\text{L}}\text{arg}\,(1+\frac{z}{m})^{m+1}=\text{arg}\,f(z).$$

Therefore, from the preceding inequalities, since $\text{mod}\,(1+\frac{z}{n})^n$ lies between two numbers, both of which have the same limit $\text{mod}\,f(z)$

$$\underset{n\doteq\infty}{\text{L}}\text{mod}\,(1+\frac{z}{n})^n=\text{mod}\,f(z).$$

Similarly, $\underset{n\doteq\infty}{\text{L}}\text{arg}\,(1+\frac{z}{n})^n=\text{arg}\,f(z)$.

Therefore, $\underset{n\doteq\infty}{\text{L}}(1+\frac{z}{n})^n=f(z)$.

§4. CASE 2. $180° > \mathrm{ARG}\, z > 90°$.

With the same notation as before:

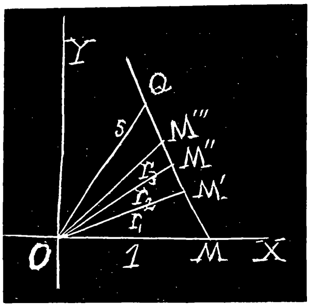

Fig. 2.

$$\phi_1 < \phi_2 < \phi_3, \quad m\phi_1 < n\phi_2 < (m+1)\phi_3,$$

and as before:

$$\mathop{L}_{n \doteq \infty} \arg\left(1+\frac{z}{n}\right)^n = \arg f(z).$$

But we can not be sure that the inequalities (1) hold. We can however proceed as follows: By Trigonometry, Fig. 2, where length $MQ = \frac{r}{\lambda}$,

$$\left[\mathrm{mod}\left(1+\frac{z}{\lambda}\right)\right]^2 = s^2 = 1^2 + \left(\frac{r}{\lambda}\right)^2 + \frac{2r\cos\phi}{\lambda} = 1 + \frac{r}{p}, \text{ where } p = \frac{\lambda}{2\cos\phi + r/\lambda}.$$

$\left[\mathrm{mod}\left(1+\frac{z}{\lambda}\right)^\sigma\right]^2 = \left[\left(1+\frac{r}{p}\right)^p\right]^{\sigma/p}$. If now σ and p are two real numbers which become indefinitely great in any way such that the difference between σ and λ remains finite then we have:

$$\mathop{L}_{\lambda \doteq \infty}\left(\frac{\sigma}{\lambda}\right) = 1. \quad p \doteq \infty, \text{ as } \lambda \doteq \infty. \quad \mathop{L}_{\lambda \doteq \infty} \frac{\sigma}{p} = 2\cos\phi.$$

$$\mathop{L}_{\lambda \doteq \infty}\left[\mathrm{mod}\left(1+\frac{z}{\lambda}\right)^\sigma\right]^2 = e^{2r\cos\phi}.$$

Taking the square root we have:

$$\mathop{L}_{\lambda \doteq \infty} \mathrm{mod}\left(1+\frac{z}{\lambda}\right)^\sigma = e^{r\cos\phi} = e^{\mathrm{mod}(z)\cos\arg(z)}$$

a result that is independent of λ and σ.

As special cases, if (λ, σ) have the pairs of values $(m+1, m)$, (n, n), $(m, m+1)$, (m, m), we see that the three numbers of the inequalities (1) have the same limit, and that this limit is $\mathrm{mod} f(z)$.

Since the modulus and argument of $\left(1+\frac{z}{n}\right)^n$ have for limit, respectively, the modulus and argument of $f(z)$, we have:

$$\mathop{L}_{n \doteq \infty}\left(1+\frac{z}{n}\right)^n = f(z).$$

This method might also have been used in case 1.

§5. CASE 3. WHEN 360°>ARG z>180°.

We may fold over the figure on the axis of x, and consider the conjugate. Thus by cases 1 or 2:

$$\underset{n \doteq \infty}{\mathrm{L}} \left(1+\frac{z'}{n}\right)^n = f(z'),$$

where z' is the conjugate of z; *i. e.*,

$$\underset{n \doteq \infty}{\mathrm{L}} \left(1+\frac{x-yi}{n}\right)^n = f(x-yi).$$

As the real and the imaginary parts are respectively equal on the two sides of this equality, the equality will remain if i is changed into $-i$. Hence

$$\underset{n \doteq \infty}{\mathrm{L}} \left(1+\frac{x+yi}{n}\right)^n = f(x+yi), \text{ or } \underset{n \doteq \infty}{\mathrm{L}} \left(1+\frac{z}{n}\right)^n = f(z).$$

§6. THE EXTENSION TO A NEGATIVE n.

Let $n=-p$, then

$$\underset{n \doteq \infty}{\mathrm{L}} \left(1+\frac{z}{n}\right)^n = \underset{p \doteq \infty}{\mathrm{L}} \left(1-\frac{z}{p}\right)^{-p} = \underset{p \doteq \infty}{\mathrm{L}} \left(1+\frac{z}{p-z}\right)^p$$

$$= \underset{p \doteq \infty}{\mathrm{L}} \left(1+\frac{z}{\rho(\cos\alpha+i\sin\alpha)}\right)^p = \underset{p \doteq \infty}{\mathrm{L}} \left[\left(1+\frac{z(\cos\alpha-i\sin\alpha)}{\rho}\right)^\rho\right]^{p/\rho}$$

$$= f\,[z\mathrm{L}(\cos\alpha-i\sin\alpha)] = f(z), \text{ since}$$

$$p-z = p-x-yi = \rho(\cos\alpha+i\sin\alpha), \quad \rho = [(p-x)^2+y^2]^{\frac{1}{2}},$$

$$\underset{p \doteq \infty}{\mathrm{L}} \frac{p}{\rho} = 1, \quad \underset{p \doteq \infty}{\mathrm{L}} \alpha = 0, \text{ and } \underset{p \doteq \infty}{\mathrm{L}} (\cos\alpha-i\sin\alpha) = 1.$$

Hence in all cases when z is complex and n is real

$$F(z) = f(z).$$

§7. THE CASE WHEN z AND n ARE BOTH COMPLEX.

We quote from the text p. 240: "If both z and n are complex, we have, if $n=m(\cos\phi+i\sin\phi)$,

$$\underset{n\doteq\infty}{\mathrm{L}}\left(1+\frac{z}{n}\right)^n=\underset{m\doteq\infty}{\mathrm{L}}\left(1+\frac{z}{m(\cos\phi+i\sin\phi)}\right)^{m(\cos\phi+i\sin\phi)}$$

$$=\underset{m\doteq\infty}{\mathrm{L}}\left[\left(1+\frac{z(\cos\phi-i\sin\phi)}{m}\right)^m\right]^{\cos\phi+i\sin\phi}=\{f[z(\cos\phi-i\sin\phi)]\}^{\cos\phi+i\sin\phi}.$$

This is as far as we can carry the proof. If, however, we agree to give to a complex exponent such an interpretation that the third property, viz: $f(z)=[f(1)]^z$, shall still hold even when z is complex, we have

$$F(z)=\{f[z(\cos\phi-i\sin\phi)]\}^{\cos\phi+i\sin\phi}=f(1)^z=f(z).$$

Thus for all values of x and n, we have

$$F(x)=f(x).$$

That the series denoted by $f(x)$ is convergent has been seen from the mode of its derivation, since each of the constituent series of which it is composed is convergent whether x be real or complex. The result

$$\underset{n\doteq\infty}{\mathrm{L}}\left(1+\frac{x}{n}\right)^n=e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$$

is known as the Exponential Theorem."

SYRACUSE UNIVERSITY, *12 April, 1909.*